Appendix B: CRPS for censored, shifted gamma distribution

Two integration formulae

Denote by F_k the CDF and by f_k the density of a gamma distribution with shape parameter k and unit scale. Further denote by $\Gamma(\cdot)$ the gamma function and by $B(\cdot,\cdot)$ is the beta function.

1. Using the relation $xf_k(x) = kf_{k+1}(x)$ we obtain

$$\int_{\tilde{y}}^{\infty} xf_k(x)\,dx = k\big(1 - F_{k+1}(\tilde{y})\big)$$

2. Using the recurrence relation

$$F_{k+1}(x) = F_k(x) - f_{k+1}(x)$$

(this can be derived for the lower incomplete gamma function via integration by parts and then be rewritten for the gamma CDF) and the duplication formula

$$\Gamma(k)\Gamma\left(k + \tfrac{1}{2}\right) = 2^{1-2k}\,\sqrt{\pi}\,\Gamma(2k),$$

we can perform integration by substitution and use the properties of the gamma density function to obtain

$$\begin{aligned}
\int_{\tilde{c}}^{\infty} F_{k+1}(x)f_k(x)\,dx &= \int_{\tilde{c}}^{\infty} F_k(x)f_k(x)\,dx - \int_{\tilde{c}}^{\infty} f_{k+1}(x)f_k(x)\,dx \\
&= \int_{F_k(\tilde{c})}^{1} x\,dx - \frac{\Gamma(2k)}{\Gamma(k)\Gamma(k+1)}\frac{1}{2^{2k}}\int_{2\tilde{c}}^{\infty} f_{2k}(x)\,dx \\
&= \frac{1}{2}\left(1 - F_k(\tilde{c})^2\right) - \frac{\Gamma(k+\frac{1}{2})}{2\sqrt{\pi}\,\Gamma(k+1)}\big(1 - F_{2k}(2\tilde{c})\big) \\
&= \frac{1}{2}\left(1 - F_k(\tilde{c})^2\right) - \frac{1}{2\pi}\,B(\tfrac{1}{2}, k+\tfrac{1}{2})\,\big(1 - F_{2k}(2\tilde{c})\big)
\end{aligned}$$

where the last equation uses that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Applying again the relation $xf_k(x) = kf_{k+1}(x)$ and integration by parts, we then get

$$\begin{aligned}
\int_{\tilde{c}}^{\infty} xF_k(x)f_k(x)\,dx &= k\big(1 - F_k(\tilde{c})F_{k+1}(\tilde{c})\big) - k\int_{\tilde{c}}^{\infty} F_{k+1}(x)f_k(x)\,dx \\
&= \frac{k}{2}\left(1 - 2F_k(\tilde{c})F_{k+1}(\tilde{c}) + F_k(\tilde{c})^2\right) + \frac{k}{2\pi}\,B(\tfrac{1}{2}, k+\tfrac{1}{2})\,\big(1 - F_{2k}(2\tilde{c})\big)
\end{aligned}$$

The actual CRPS formula

We use the abbreviations $\tilde{c} := -\frac{\delta}{\theta}$ and $\tilde{y} := \frac{y-\delta}{\theta}$, perform first integration by substitution, then integration by parts, and finally plug in the preceding expressions to get

$$\begin{aligned}
&\int_0^{\infty} \left|F_k\left(\tfrac{x-\delta}{\theta}\right) - \mathbf{1}_{[y,\infty)}(x)\right|^2 dx \;=\; \theta\int_{\tilde{c}}^{\infty} \left|F_k(x) - \mathbf{1}_{[\tilde{y},\infty)}(x)\right|^2 dx \\
&\quad= \theta\int_{\tilde{c}}^{\tilde{y}} F_k(x)^2\,dx \;+\; \theta\int_{\tilde{y}}^{\infty} \left(F_k(x) - 1\right)^2 dx \\
&\quad= \theta\tilde{y}\left(2F_k(\tilde{y}) - 1\right) - \theta\tilde{c}\,F_k(\tilde{c})^2 \;-\; 2\theta\int_{\tilde{c}}^{\infty} xF_k(x)f_k(x)\,dx \;+\; 2\theta\int_{\tilde{c}}^{\infty} xf_k(x)\,dx \\
&\quad= \theta\tilde{y}\left(2F_k(\tilde{y}) - 1\right) \;-\; \theta\tilde{c}\,F_k(\tilde{c})^2 \;+\; \theta k\left(1 + 2F_k(\tilde{c})F_{k+1}(\tilde{c}) - F_k(\tilde{c})^2 - 2F_{k+1}(\tilde{y})\right) \\
&\qquad - \frac{\theta k}{\pi}\,B\left(\tfrac{1}{2}, k+\tfrac{1}{2}\right)\left(1 - F_{2k}(2\tilde{c})\right)
\end{aligned}$$